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LETTER TO THE EDITOR

General self-dual spin models in two dimensions

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Abstract. We show that a very large class of two-dimensional $Z(N)$ spin models with multi-spin interactions is self-dual. By using Griffiths-like inequalities on correlation functions we expect that all these theories, for N sufficiently large, will show an intermediate disordered massless phase (infinite correlation length) between massive high- and low-temperature phases.

Several spin systems with $Z(N)$ symmetry in two dimensions (Elitzur *et al* 1979, Cardy 1980, Alcaraz and Köberle 1980, 1981, Alcaraz and Jacobs 1982a, b), three dimensions (Amit *et al* 1981, Alcaraz *et al* 1982 b, c) and $Z(N)$ gauge invariant models in four dimensions have the same structure under duality transformation (Savit 1980 and references therein), with the Villain as well as the Potts version of those models being self-dual for all N. This self-duality property (in the Villain version) enables one to locate the critical temperature (Kramers and Wannier 1941) if a uniqueness assumption for the phase transition holds. By using Griffiths-like correlation inequalities and the fact that the U(1) (limit when $N \to \infty$) model has a phase transition at non-zero temperature, it is possible to show that for N sufficiently large those theories show two phase transitions with the intermediate phase being massless (infinite correlation length) (Elitzur *et al* 1979, Ukawa *et al* 1980, Amit *et al* 1981, Alcaraz *et al* 1982a, c). Therefore it is interesting to obtain a general model that possesses the same duality structure as the above models. We present in this letter a large class of two-dimensional spin models that exhibit this self-duality structure.

We consider a square lattice and define a $Z(N)$ spin variable $S(\boldsymbol{r}) = \exp[i(2\pi/N)q(\boldsymbol{r})]$, $(q(\boldsymbol{r}) = 0, 1, \ldots, N-1)$ at each lattice point, periodic boundary conditions being assumed. We shall consider first, for simplicity, the model in which there is an m-spin interaction in the x direction and an n-spin interaction in the y direction (even more general models will be discussed later). For a Hamiltonian (or action) to possess the $Z(N)$ symmetry it must be a function of the cluster variables

$$S_c^x(\boldsymbol{r}) = \prod_{l=0}^{m-1} S(\boldsymbol{r} + \hat{i}l) = \exp i \frac{2\pi}{N} \sum_{l=0}^{m-1} q(\boldsymbol{r} + \hat{i}l) = \exp i \frac{2\pi}{N} q_c^x(\boldsymbol{r}), \tag{1a}$$

$$S_c^y(\boldsymbol{r}) = \prod_{l=0}^{n-1} S(\boldsymbol{r} + \hat{j}l) = \exp i \frac{2\pi}{N} \sum_{l=0}^{n-1} q(\boldsymbol{r} + \hat{j}l) = \exp i \frac{2\pi}{N} q_c^y(\boldsymbol{r}). \tag{1b}$$

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We can then define the general Hamiltonian (Cardy 1980, Alcaraz and Köberle 1980, 1981)

$$\mathcal{H} = -\sum_{\alpha=1}^{[N/2]} \left(J_\alpha^x \cos \frac{2\pi}{N} \alpha q_c^x + J_\alpha^y \cos \frac{2\pi}{N} \alpha q_c^y \right), \tag{2}$$

$[N/2]$ being the integer part of $N/2$. It is interesting to observe that this model has non-local $Z(N)$ symmetry involving a fraction $4/nm$ of the total number of spins, producing then a $N^{(n-1)(m-1)}$-fold degeneracy of the ground state. The Potts version (Hamiltonian analogue to the Potts model) of the above model corresponds to $J_1 = J_2 = \ldots = \frac{1}{2}(1+(-)^N) J_{[N/2]}$ and the clock version corresponds to $J_\alpha = J\delta_{\alpha,1}$. The case $m = n = 2$ reduces to the general self-dual $Z(N)$ two-body spin model (Cardy 1980, Alcaraz and Köberle 1980, 1981). The Potts version has been shown to be self-dual (Turban and Debierre 1982, Turban 1982). We will show that the general model (2) has the same duality structure already known for the two-body interaction case.

For simplicity we will make the duality transformation in the periodic Gaussian or Villain form of the clock model, and the duality for the general model follows in a straightforward fashion. By standard techniques (José *et al* 1977, Savit 1980) the Villain approximation is introduced by defining a two-component integer field $\boldsymbol{J}(r) \equiv (J^x(r), J^y(r))$ $(-\infty < J^i(r) < \infty)$ at each lattice point. The partition function is then given by

$$Z^V = \prod_{\{\boldsymbol{J}(r)=-\infty\}}^{\infty} \prod_{\{q(r)=0\}}^{N-1} \exp -\frac{\beta}{2} \sum_r |\boldsymbol{q}_c(r) 2\pi/N - 2\pi \boldsymbol{J}(r)|^2 \tag{3}$$

where the first and second sums are the trace over the $\boldsymbol{J}(r)$ and $q(r)$ field and the sum in the exponent extends over all lattice points. For brevity of notation we have defined

$$\boldsymbol{q}_c(r) \equiv (q_c^x(r), q_c^y(r)). \tag{4}$$

By use of the Poisson summation formulae we can write

$$\sum_{\boldsymbol{J}(r)} \exp[-\tfrac{1}{2}\beta |2\pi/N \boldsymbol{q}_c(r) - 2\pi \boldsymbol{J}(r)|^2] = \frac{1}{(2\pi\beta)^{1/2}} \sum_{\boldsymbol{l}(r)=-\infty}^{\infty} \exp\left(-\frac{l^2(r)}{2\beta} + \mathrm{i} \frac{2\pi}{N} \boldsymbol{l}(r) \cdot \boldsymbol{q}_c(r) \right) \tag{5}$$

where $\boldsymbol{l}(r) \equiv (l^x(r), l^y(r))$ is a two-component integer field defined on sites. It is convenient to write the integer field $\boldsymbol{l}(r)$ as a sum of two other fields $\boldsymbol{l}(r) = N\boldsymbol{\rho}(r) + \boldsymbol{\nu}(r)$ with $-\infty \leqslant \rho^i(r) \leqslant \infty$, $0 \leqslant \nu^i(r) \leqslant N-1$, $(i = 1, 2)$, so that the partition function takes the form

$$Z = C(\beta) \sum_{\{\boldsymbol{\rho}(r)=-\infty\}}^{\infty} \sum_{\{\boldsymbol{\nu}(r)=0\}}^{N-1} \exp -\frac{1}{2\beta} \sum_r |N\boldsymbol{\rho}(r) + \boldsymbol{\nu}(r)| \sum_{\{q(r)=0\}}^{N-1} \exp \mathrm{i} \frac{2\pi}{N} \boldsymbol{\nu}(r) \cdot \boldsymbol{q}_c(r) \tag{6}$$

where $C(\beta)$ is a harmless constant. In order to perform the $q(r)$-summations we must isolate the $q(r)$ variables (integrate by parts); this may be done by writing

$$\sum_r \boldsymbol{\nu}(r) \cdot \boldsymbol{q}_c(r) = \sum_r q(r) \nu_c^*(r) \tag{7}$$

in which we have introduced the scalar integer (mod N) field

$$\nu_c^*(r) = \sum_{l=0}^{m-1} \nu^x(r - l\hat{i}) + \sum_{l=0}^{n-1} \nu^y(r - l\hat{j}) \qquad (\mathrm{mod}\, N) \tag{8}$$

so that by inserting equation (7) in equation (6) and performing the $q(r)$-summations we get

$$Z = C(\beta) \sum_{\{\rho(r)=-\infty\}}^{\infty} \sum_{\{\nu(r)=0\}} \exp - \sum_{r} \frac{1}{2\beta} |N\rho(r) + \nu(r)|^2 \prod_{r} N\delta_k[\nu_c^*(r)] \qquad (9)$$

where δ_k is a Kronecker delta function (since $\nu_c^*(r)$ is defined modulo N). In order to satisfy the δ_k-requirements, that is $\nu_c^*(r) = 0$ for all r, we define in the dual lattice (the square lattice whose lattice points are given by $\underline{r} = r + \frac{1}{2}(\hat{i} + \hat{j})$, and with periodic boundary conditions) the dual field $0 \leq \tilde{q}(\underline{r}) \leq N-1$. We can verify easily that if

$$\nu^y(r) = \sum_{l=0}^{m-1} \tilde{q}(\underline{r} - l\hat{i}) = \tilde{q}_c^x[\underline{r} - (m-1)\hat{i}] \qquad (\mathrm{mod}\, N), \qquad (10a)$$

$$\nu^x(r) = -\sum_{l=0}^{n-1} \tilde{q}(\underline{r} - l\hat{j}) = -\tilde{q}_c^y[\underline{r} - (n-1)\hat{j}] \qquad (\mathrm{mod}\, N), \qquad (10b)$$

the δ_k-requirements in equation (9) are satisfied. Conversely, given a $\nu(r)$ configuration that satisfies the δ_k-requirements, we can find the corresponding dual field configuration

$$\tilde{q}(\underline{r}) = \sum_{l=0}^{\infty} [\nu^y(r - ml\hat{i}) - \nu^y(r - ml - 1)\hat{i}]. \qquad (11)$$

Of course this infinite string has a sort of gauge symmetry in which we can add $\nu_c^*(r)$ at any point r, modifying the string path. Clearly there are many configurations of $\tilde{q}(\underline{r})$ variables (related by Z(N) symmetry) which produce the same $\nu(r)$ configuration; however, this will generate a uniform overcounting, giving a harmless overall infinite numerical factor (Savit 1980). Finally, making the convenient redefinitions

$$\tilde{\rho}^y(\underline{r} - (n-1)\hat{j}) \equiv \rho^x(r), \qquad \tilde{\rho}^x(\underline{r} - (m-1)\hat{i}) \equiv -\rho^y(r), \qquad (12a, b)$$

the partition function takes the form

$$Z = D(\beta) \sum_{\{\tilde{\rho}(\underline{r})=-\infty\}}^{\infty} \sum_{\{\tilde{q}_c(\underline{r})=0\}}^{N-1} \exp - \left(\frac{N^2}{2\beta(2\pi)^2} |2\pi/N\tilde{q}_c(\underline{r}) - 2\pi\tilde{\rho}(\underline{r})|^2 \right) \qquad (13)$$

so that

$$Z(\beta) = D(\beta) Z(\tilde{\beta}) \qquad (14)$$

where $D(\beta)$ is a harmless constant and $\tilde{\beta} \underset{W}{=} N^2/[\beta(2\pi)^2]$. Therefore, the Villain model is self-dual for all N. If the Villain model has a unique phase transition it must occur at the self-dual point $\beta^* = N/2\pi$.

Performing the duality transformation for the general model defined by equation (2), we obtain exactly the same duality structure known for the general Z(N), $m = n = 2$ spin model (Cardy 1980, Alcaraz and Köberle 1980, 1981); for example, all the Potts models ($J_1 = J_2 = \ldots = J_{[N/2]}[1 + (-)^N]/2$) are self-dual, and the vector models ($J_\alpha = J_1 \delta_{\alpha 1}$) are self-dual when $N \leq 4$, etc.

We may prove this duality structure for even more general spin systems than those given by the Hamiltonian (2). Following the general work of Wegner (1971) and Savit (1982), what is fundamental for self-duality is to have the existence of one field (variable) and two types of potentials for each lattice point. We can therefore define general clusters $q_c = (q_c^x, q_c^y)$ (not necessarily straight clusters), and it is possible to find the dual clusters $\tilde{q}_c(\underline{r})$ (see equation (10)) that solves the δ_k-requirements (see equation (9)). The dual cluster $\tilde{q}_c^x(\tilde{q}_c^y)$ has the same form, rotated by π, as the original cluster $q_c^x(q_c^y)$; in figure 1 we show two examples of clusters with their duals. In general

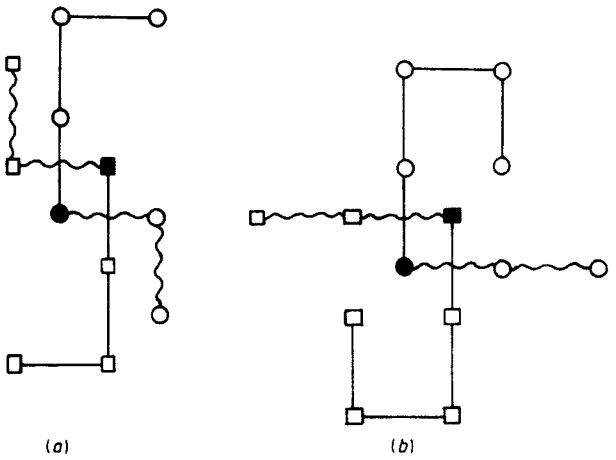

Figure 1. The circles (squares) are the original (dual) lattice points. The sum of $q(r)$ ($\tilde{q}(\underline{r})$) variables that lie in the straight and wavy segments in the original (dual) lattice corresponds to the cluster variables q_c^y and q_c^x ($\tilde{q}_c^y$ and $\tilde{q}_c^x$) respectively.

these arbitrary clusters will produce self-dual models with somewhat complicated symmetries. There is however a large family of clusters (in figure 2 we show some examples) whose Hamiltonians exhibit a non-local Z(N) symmetry of the same nature as that of straight clusters. Let us stress that the clusters given by figure 2(c) produce the triangular Z(N) model with three-body interaction, which is already known to be self-dual (Alcaraz and Jacobs 1982a, b). We may even consider models in which the clusters associated with each point are not equal. That is, if we divide the square

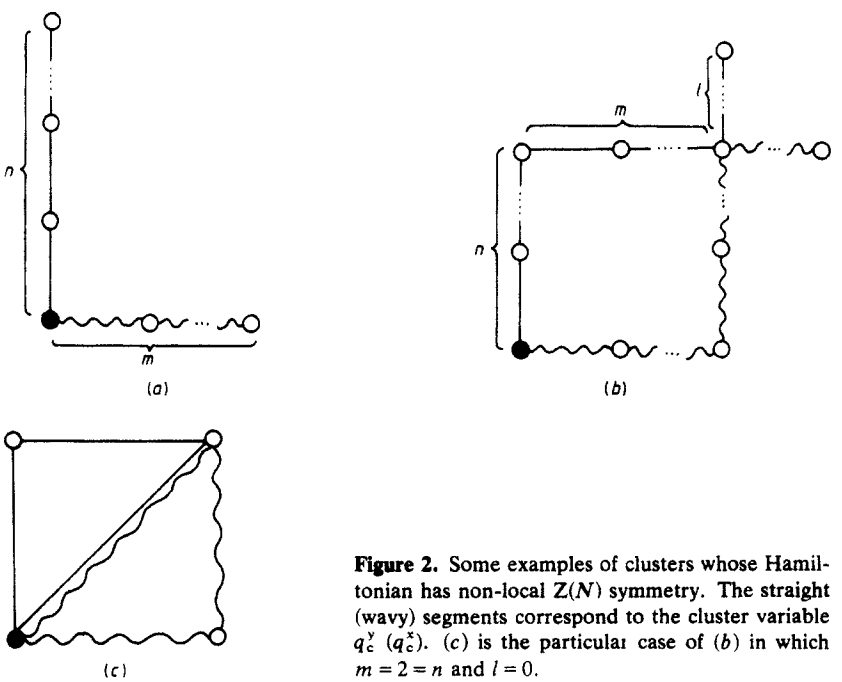

Figure 2. Some examples of clusters whose Hamiltonian has non-local Z(N) symmetry. The straight (wavy) segments correspond to the cluster variable q_c^y (q_c^x). (c) is the particular case of (b) in which $m = 2 = n$ and $l = 0$.

lattice into l square sublattices it is possible to define clusters attached to each sublattice in a different way, but preserving the self-dual structure. One example is the case when we have two sublattices and the clusters are shown in figure 3; it is easy to verify that those clusters generate the Z(N)-triplet Union-Jack model which is known to be self-dual (Alcaraz and Cardy 1982).

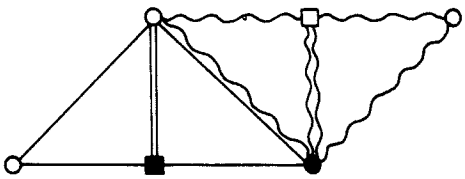

Figure 3. The circles and squares are lattice points of different sublattices. The straight and wavy segments correspond to the cluster variables attached to different sublattices.

By standard techniques (Elitzur *et al* 1979, Amit *et al* 1981, Alcazar *et al* 1982a, c) it is possible to prove Griffiths-like inequalities on correlation functions for those models (details will be presented elsewhere). Defining the M-spin (order variable) correlation function

$$C_N(r_1, r_2, \ldots, r_M) \equiv \left\langle \cos\left(\frac{2\pi}{N} \sum_{i=1}^{M} q(r_i)\right)\right\rangle \tag{15a}$$

and the M-point correlation function for the dual variable

$$D_N(\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_m) \equiv \left\langle \cos\left(\frac{2\pi}{N} \sum_{i=1}^{m} \tilde{q}(\underline{r}_i)\right)\right\rangle \tag{15b}$$

and by using the Villain version of the theory it is possible to prove the inequalities

$$C_N^{\mathrm{V}}(r_1, \ldots, r_m) \geqslant C_\infty^{\mathrm{V}}(\underline{r}_1, \ldots, \underline{r}_m), \qquad D_N^{\mathrm{V}}(r_1, \ldots, r_m) \leqslant D_\infty^{\mathrm{V}}(\underline{r}_1, \ldots, \underline{r}_m). \tag{16a, b}$$

Therefore by using standard arguments (Elitzur *et al* 1979) these inequalities imply the existence of an intermediate phase (with the full symmetry of the Hamiltonian) between the low- and high-temperature phases for N sufficiently large if the U(1) Villain theory has a phase transition at finite temperature. By studying the Coulomb gas representation (details will be reported elsewhere) of the Villain formulation of these theories with non-local symmetry U(1) (see for example figure 2), we verify that they may be represented by a gas of complex charges interacting logarithmically, and hence using the Kosterlitz and Thouless (1973) arguments we expect that these U(1) theories should have a phase transition at a finite inverse temperature β_k^{c} (that would depend on the type of cluster), with the low-temperature phase disordered (Mermin and Wagner 1966) and massless (infinite correlation length). Therefore we conclude this letter by stating that all of these large clan of models with non-local Z(N) symmetry studied here are expected, in the Villain formulation, to have an intermediate massless phase for $N > N_{\mathrm{c}} = 2\pi\beta_k^{\mathrm{c}}$; furthermore the general Hamiltonian defined by equation (2) will have (assuming the continuity of the critical surface in the J_α parameter space) massless phases for $N > N_{\mathrm{c}}$.

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